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Higher-order gauge invariant Lagrangians on T^*M

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Abstract. Higher-order Lagrangians on $T^*(M)$ invariant under the natural representation of gauge fields of $M \times U(1)$ on the cotangent bundle are determined.

1. Introduction

Let $\pi : P \to M$ be a G-principal bundle. The geometric interpretation of Utiyama's theorem ([4, 18]) states that a Lagrangian $\mathcal{L}$ on the 1-jet bundle of the connections of P is invariant under the natural representation of the gauge algebra of P on connections, if and only if $\mathcal{L}$ factors through the curvature mapping κ (this is the map sending each connection Γ onto its curvature form $\Omega_\Gamma = \mathrm{d}\omega_\Gamma + [\omega_\Gamma, \omega_\Gamma]$) as follows $\mathcal{L} = \bar{\mathcal{L}} \circ \kappa$, where $\bar{\mathcal{L}}$ is a function on the curvature bundle—viewed as a zero-order Lagrangian—which in turn must be invariant under the gauge algebra representation. In the Abelian case the above statement is particularly simple since $\bar{\mathcal{L}}$ automatically becomes gauge invariant as the adjoint representation of the Abelian group G is trivial and hence the theorem simply states that the Lagrangian is invariant if and only if it factors through the curvature mapping, which is nothing but the exterior differential of ω_Γ in this case.

The aim of this paper is to extend the above result to higher-order Lagrangians. Without doubt, in field theory the most important Abelian group is $U(1)$, so we confine ourselves to considering the trivial bundle $M \times U(1)$, where the manifold M can be understood to be a spacetime. However, we do not make any hypothesis on it, assuming throughout that M is an arbitrary n-dimensional C^∞ manifold.

2. Gauge algebra representation on $J^r(T^*M)$

2.1. Introducing the coordinate systems

Let $(N; q_1, \dots, q_n)$ be an open coordinate domain in M. As usual, let us denote by (q_i, p_i) the coordinate system induced on $(p^1)^{-1}(N)$ from $(N; q_1, \dots, q_n)$, $p^1 : T^*M \to M$ being the canonical projection; i.e.

$$w = \sum_{i=1}^{n} p_i(w)(\mathrm{d}q_i)_x \qquad x \in M$$

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for every covector $w \in T^*_x(M)$, where $\mathrm{d}f$ denotes the differential of a function $f \in C^\infty(M)$. We parametrize the points in $U(1)$ as $z = \exp(\mathrm{i}t)$, so that $(q_1, \ldots, q_n; t)$ is a coordinate system for $M \times U(1)$. Let us denote by (q_j, p^i_α), $1 \leqslant i, j \leqslant n$, $0 \leqslant |\alpha| \leqslant r$, $p^i_0 = p_i$, the coordinate system induced on $(p^1)^{-1}_r(N)$, where $(p^1)_r : J^r(T^*M) \to M$ stands for the canonical projection; i.e.

$$p^i_\alpha(j^r_x\omega) = \frac{\partial^{|\alpha|}(p_i \circ \omega)}{\partial q_1^{\alpha_1} \ldots \partial q_n^{\alpha_n}}(x) \tag{1}$$

for every r-jet of differential form $j^r_x\omega \in J^r(T^*M)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. More generally, if $p^k : \bigwedge^k T^*M \to M$ is the canonical projection and $(q_j, p_{i_1,\ldots,i_k})$, $1 \leqslant j, i_1 < \cdots < i_k \leqslant n$, stands for the induced coordinate system on $(p^k)^{-1}(N)$, that is,

$$w_k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} p_{i_1,\ldots,i_k}(w_k)(\mathrm{d}q_{i_1})_x \wedge \cdots \wedge (\mathrm{d}q_{i_k})_x \tag{2}$$

for every k-covector $w_k \in \wedge^k T^*_x M$, then we denote by

$$(q_j, p^{i_1,\ldots,i_k}_\alpha) \qquad 1 \leqslant j, i_1 < \cdots < i_k \leqslant n \qquad 0 \leqslant |\alpha| \leqslant r$$

the induced coordinate system on $(p^k)^{-1}_r(N)$, defined as in formula (1). Basically, we are concerned with the cases $k = 1, 2$.

2.2. Gauge fields on $M \times U(1)$

As is well known (e.g. [4(3.2), 8(III.35), 13(3.1)]), gauge transformations on $M \times U(1)$ can be identified with the automorphisms of $M \times U(1)$ that induce the identity over M. More precisely, an automorphism of $M \times U(1)$ is a diffeomorphism $\Phi : M \times U(1) \to M \times U(1)$ such that $\Phi(u \cdot g) = \Phi(u) \cdot g$, $\forall u \in M \times U(1)$, $\forall g \in U(1)$. This implies that Φ maps each fibre of $\mathrm{pr}_1 : M \times U(1) \to M$ onto another fibre and hence there exists a unique diffeomorphism ϕ making the following diagram commutative:

$$\begin{array}{ccc} M \times U(1) & \xrightarrow{\Phi} & M \times U(1) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow{\phi} & M \end{array}$$

If ϕ is the identity map of M, then we say that Φ is a bundle automorphism or a gauge transformation and this means that Φ transforms each fibre of $M \times U(1)$ onto itself. It is not difficult to see that bundle automorphisms are locally parametrized as $\Phi(x, t) = (x, t + \psi(x))$, for a differentiable function $\psi : N \to \mathbb{R}$ (cf [9, III.B]). Let us denote by $\mathrm{Gau}(M \times U(1))$ the gauge group of the bundle; i.e. the group of all bundle automorphisms. Geometrically, the *gauge algebra* of an arbitrary principal bundle is introduced as the 'infinitesimal version' of its gauge group (cf [4(3.2.9–3.2.17), 8(III.35 p 278)]). This means that a vector field belongs to the gauge algebra of the bundle if and only if the flow that it generates belongs to the gauge group; that is, it consists of gauge transformations. Let Φ_s, $s \in \mathbb{R}$, be the flow of a vector field $X \in \mathfrak{X}(M \times U(1))$. Then, it is quickly checked that $\Phi_s \in \mathrm{Gau}(M \times U(1))$, $\forall s$ (that is, X belongs to the gauge algebra) if and only if:

(1) X is a $U(1)$-invariant, i.e. $R_z \cdot X = X$, $\forall z \in U(1)$, where R_z stands for the translation by the element z; and

(2) X is a pr_1-vertical vector field on $M \times U(1)$; i.e. $(\mathrm{pr}_1)_* X = 0$.

Accordingly, we denote the vector fields satisfying (1) and (2) by $\mathrm{gau}(M\times U(1))$. Also, it is easy to prove (e.g. see [9]) that a vector field X on $M\times U(1)$ belongs to $\mathrm{gau}(M\times U(1))$ if and only if it can be locally written as

$$X = g(q_1,\ldots,q_n)\frac{\partial}{\partial t} \qquad g\in C^\infty(N). \tag{3}$$

2.3. The bundle of connections of $M\times U(1)$ and T^*M

Let $\pi : P \to M$ be an arbitrary principal bundle of structure group G. The group G acts on the tangent bundle of P in a natural way. Set $Q = T(P)/G$. Then, Q is endowed with a natural vector bundle structure over M, whose sections can be identified with the G-invariant vector fields of P, and we have the following exact sequence of vector bundles over M, the so-called *Atiyah sequence* (cf [2, Theorem 1]):

$$0\to L(P)\to Q \xrightarrow{\pi_*} T(M)\to 0$$

where $L(P)$ stands for the adjoint bundle; i.e. the bundle associated to P by the adjoint representation of G on its Lie algebra. Given a connection Γ on P, the horizontal lift X^* of a vector field X on M is a G-invariant vector field π-projectable onto X (cf [10, II. Proposition 1.2]). Hence, we can define a section $\sigma_\Gamma : T(M)\to Q$ of the projection $\pi_* : Q\to T(M)$ in the Atiyah sequence by setting $\sigma_\Gamma(X) = X^*$, which completely determines the connection Γ. Moreover, any such section is induced by the horizontal lifting of a connection on P; in other words, the connections on a principal bundle can be identified with the splittings of the Atiyah sequence of the given bundle (cf [2, p 188]). In this way we can construct a fibre bundle $p : \mathcal{C}(P)\to M$ (the bundle of connections) whose global sections coincide with the connections on P, as follows.

The fibre of $\mathcal{C}(P)$ over a point $x\in M$ consists of all linear maps $s : T_x(M)\to Q_x$ such that $\pi_*\circ s$ is the identity map of the tangent space $T_x(M)$. If s' is another element in $\mathcal{C}(P)_x$, then for every tangent vector $X\in T_x(M)$ we have

$$\pi_*((s'-s)(X)) = \pi_*(s'(X)) - \pi_*(s(X)) = X - X = 0$$

and accordingly, $(s'-s)(X)\in\ker\pi_* = L(P)$; or, equivalently, $s'-s\in T_x^*(M)\otimes L(P)_x$. Furthermore, if we add a homomorphism $h\in T_x^*(M)\otimes L(P)_x$ to an element $s\in\mathcal{C}(P)_x$, we obtain another element $s+h$ in $\mathcal{C}(P)_x$. Because of this, we say that $\mathcal{C}(P)$ is an affine bundle modelled over the vector bundle $T^*(M)\otimes L(P)$. For the details of this construction we refer the reader to [7]. By virtue of the properties of the horizontal lifting, every connection Γ determines a section of $p : \mathcal{C}(P)\to M$ by the formula $x\mapsto\sigma_\Gamma(x)$, and conversely.

The cotangent bundle occurs in this theory as it can be canonically identified with the bundle of connections of the trivial $U(1)$-principal bundle $\mathrm{pr}_1 : M\times U(1)\to M$; i.e. we have a natural isomorphism of fibre bundles

$$\mathcal{C}(M\times U(1))\cong T^*(M).$$

In fact, let A be the standard basis of the Lie algebra of $U(1)$; i.e. A is the vector of $u(1)$ determined by the homomorphism $\mathbb{R}\to U(1)$, $t\mapsto\exp(it)$. Then, every connection of the form ω_Γ can be uniquely written as $\omega_\Gamma = (\mathrm{d}t + (\mathrm{pr}_1)^*\omega)\otimes A$, where ω is an arbitrary one-form on M, and the isomorphism between the bundle of connections of $M\times U(1)$ and T^*M is stated by setting $\omega_\Gamma\leftrightarrow\omega$.

2.4. Representing gau$(M \times U(1))$ in $\mathfrak{X}(J^r(T^*M))$

Each bundle automorphism $\Phi : M \times U(1) \to M \times U(1)$ acts on the connections of this bundle by the rule $\omega_{\Phi\cdot\Gamma} = (\Phi^{-1})^*\omega_\Gamma$([10, II, Proposition 6.1(a) and (b)]). Taking into account the identification between the bundle of connections of $M \times U(1)$ and T^*M, the above action induces a diffeomorphism of the cotangent bundle $\tilde{\Phi} : T^*M \to T^*M$, such that $p^1 \circ \tilde{\Phi} = p^1$. In this way we obtain a group homomorphism $\mathrm{Gau}(M \times U(1)) \to \mathrm{Diff}(T^*M)$. Let Φ_t be the local flow of a vector field $X \in \mathrm{gau}(M \times U(1))$. Then, $\tilde{\Phi}_t$ is the local flow of a vector field $\tilde{X} \in \mathfrak{X}(T^*M)$, and the mapping $X \mapsto \tilde{X}$ establishes a representation of $\mathrm{gau}(M \times U(1))$ into $\mathfrak{X}(T^*M)$, whose local expression is given by ([9])

$$\tilde{X} = -\sum_{i=1}^{n} \frac{\partial g}{\partial q_i} \frac{\partial}{\partial p_i} \tag{4}$$

if the local expression for X is that of formula (3). In other words, the natural representation of the gauge algebra on the bundle of connections of $M \times U(1)$ coincides with the Hamiltonian vector field associated to the function g in formula (3) with respect to the exterior differential of Liouville's form on the cotangent bundle.

The general formulae of vector field prolongation by infinitesimal contact transformation to jet bundles (see [12] or [14]) and formula (4) above, then provide the following local expression of the r-jet prolongation of the vector field $\tilde{X}$, denoted by $\tilde{X}_{(r)}$:

$$\tilde{X}_{(r)} = -\sum_{i=1}^{n} \sum_{|\alpha|=0}^{r} \frac{\partial^{|\alpha|+1} g}{\partial q^{\alpha+(i)}} \frac{\partial}{\partial p^i_\alpha} \tag{5}$$

where (i) stands for the multi-index $(i) = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$.

3. Classification of invariant Lagrangians

3.1. Invariant Lagrangians on $J^r(T^*M)$

In the geometric formulation of the Lagrangian formalism for variational problems defined on a fibered manifold $p : N \to M$, the notion of a rth-order Lagrangian is introduced as an arbitrary differentiable function $\mathcal{L} : J^r(N) \to \mathbb{R}$ defined on the r-jet bundle of local sections of $p : N \to M$ (e.g. see [1, 11, 8, (IV.51)]). Therefore, in our case a rth-order Lagrangian is nothing other than a differentiable function $\mathcal{L} : J^r(T^*M) \to \mathbb{R}$. Locally, $\mathcal{L}$ can be thus expressed as a differentiable function of the coordinates (q_j, p^i_α), $1 \leqslant i, j \leqslant n$, $0 \leqslant |\alpha| \leqslant r$, $p^i_0 = p_i$, defined in section 2.1; or in other words, $\mathcal{L}$ depends on the coordinates of a point in the ground manifold M and on the partial derivatives up to order r of the components of a generic one-form on M. Such a Lagrangian is said to be *gauge invariant* if for every vector field $X \in \mathrm{gau}(M \times U(1))$, we have

$$\tilde{X}_{(r)}(\mathcal{L}) = -\sum_{i=1}^{n} \sum_{|\alpha|=0}^{r} \frac{\partial^{|\alpha|+1} g}{\partial q^{\alpha+(i)}} \frac{\partial \mathcal{L}}{\partial p^i_\alpha} = 0.$$

From this formula we thus quickly obtain the following proposition.

Proposition 1. A rth-order Lagrangian $\mathcal{L}$ on T^*M is gauge invariant if and only if it satisfies the following equations:

$$\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial p^i_{\beta-(i)}} = 0 \qquad \forall \beta \in \mathbb{N}^n, 1 \leqslant |\beta| \leqslant r+1. \tag{6}$$

In particular, for $|\beta| = 1, 2$, we obtain $\partial\mathcal{L}/\partial p_i = 0$, $\partial\mathcal{L}/\partial p^i_j + \partial\mathcal{L}/\partial p^j_i = 0$, respectively, which essentially is the contents of Utiyama's theorem in the Abelian case; that is, a first-order Lagrangian is gauge invariant if and only if it can be locally written as

$$\mathcal{L} = \bar{\mathcal{L}}(q_h, p^i_j - p^j_i), 1 \leqslant h, i < j \leqslant n.$$

Remark 1. For first-order Lagrangians the preceding definition of gauge invariance is nothing other than the standard general notion of invariance for Lagrangians defined on the connections of a G-principal bundle in the particular case $G = U(1)$ (e.g. see [4, 10.2.15]) and in fact we obtain exactly the same results. Nevertheless, the advantage of the preceding formulation lies in the fact that it allows us to generalize the notion of gauge invariance directly to arbitrary order Lagrangians in a very natural way. Let us explain this in detail. As is well known ([3, 15, 17]), the transformation law of a gauge potential on a G-principal bundle under a gauge transformation $h \in C^\infty(M, G)$ is given by $A' = h\,\mathrm{d}h^{-1} + hAh^{-1}$. In the particular case $G = U(1)$, we thus have $A' = h\,\mathrm{d}h^{-1} + A$. Writing $h(x) = \exp(\mathrm{i}\psi(x))$ and identifying A to an arbitrary one form $A = A_i\,\mathrm{d}q_i$ on M, we then obtain $A'_i = A_i + \partial\psi/\partial q_i$, or more intrinsically $\omega' = \omega + \mathrm{d}\psi$. This certainly shows that Maxwell's equations $\delta(\mathrm{d}\omega) = \delta F = 0$ remain invariant under the family of transformations $\varphi_t\omega = \omega + t\,\mathrm{d}\psi$. Moreover, probably the most simple method of obtaining Euler–Lagrangian equations which are invariant under gauge transformations is to start with a gauge invariant Lagrangian. In any case, we know that gauge invariant Lagrangians produce gauge invariant field equations. According to the previous transformation rules, the Lagrangian $\mathcal{L}(x, A_i(x), \partial A_i/\partial q_j(x))$ is gauge invariant if and only if

$$\mathcal{L}(x, A_i(x) + \partial\psi/\partial q_i(x), \partial A_i/\partial q_j(x) + \partial^2\psi/\partial q_i\partial q_j(x)) = \mathcal{L}(x, A_i(x), \partial A_i/\partial q_j(x)).$$

From the above equation, it is not difficult to prove that $\mathcal{L}$ should be independent of A_i, and should only depend on $\partial A_i/\partial q_j$ through $\partial A_i/\partial q_j - \partial A_j/\partial q_i$, which are the components of the curvature form. With the notations introduced in 2.1, notice that $p^i_j(j^1_x\omega) = \partial A_i/\partial q_j(x)$, so that the preceding result can be restated by saying that $\mathcal{L}$ is a differentiable function of q_i and $p^i_j - p^j_i$, as we did above.

3.2. The operators δ^{r-1}_k

In order to classify higher-order invariant Lagrangians we need to introduce some preliminary tools. The exterior derivative

$$\mathrm{d} : \Gamma\left(M, \bigwedge^k T^*(M)\right) \to \Gamma\left(M, \bigwedge^{k+1} T^*(M)\right)$$

is a first-order differential operator so that it factors linearly through the 1-jet extension giving rise to a homomorphism of vector bundles over M:

$$\delta_k : J^1\left(\bigwedge^k T^*M\right) \longrightarrow \bigwedge^{k+1} T^*M \tag{7}$$

i.e. δ_k is the unique homomorphism of vector bundles such that $\delta_k(j^1_x\omega_k) = (\mathrm{d}\omega_k)_x$, for every differential k-form ω_k of M defined on a neighbourhood of $x \in M$ (cf [5(14.1), 16(Théorème II.4a)]). By applying the J^{r-1} function to this map and restricting to the holonomic subbundle, we can define a homomorphism of vector bundles

$$\delta^{r-1}_k : J^r\left(\bigwedge^k T^*M\right) \to J^{r-1}\left(\bigwedge^{k+1} T^*M\right) \tag{8}$$

as being the composite

$$J^r\left(\bigwedge^k T^*M\right) \hookrightarrow J^{r-1}\left(J^1\left(\bigwedge^k T^*M\right)\right) \xrightarrow{J^{r-1}(\delta_k)} J^{r-1}\left(\bigwedge^{k+1} T^*M\right) \tag{9}$$

that is, $\delta_k^{r-1}(j_x^r\omega_k) = j_x^{r-1}(\mathrm{d}\omega_k)$. Then, we state the following 'jet version' of Poincaré's lemma.

Proposition 2. The following exact sequence of homomorphisms of vector bundles over M, holds true:

$$J^{r+1}\left(\bigwedge^{k-1} T^*M\right) \xrightarrow{\delta_{k-1}^r} J^r\left(\bigwedge^k T^*M\right) \xrightarrow{\delta_k^{r-1}} J^{r-1}\left(\bigwedge^{k+1} T^*M\right).$$

Proof. We first recall that for every vector bundle $E \to M$, we have an exact sequence of vector bundles over M:

$$0 \to S^rT^*(M) \otimes E \to J^r(E) \to J^{r-1}(E) \to 0. \tag{10}$$

Letting $E = \bigwedge^{k-1} T^*M$, we prove that the restriction of δ_{k-1}^r to the subbundle $S^{r+1}T^*(M) \otimes \bigwedge^{k-1} T^*M \subset J^{r+1}(\bigwedge^{k-1} T^*M)$ coincides with the Spencer operator (cf [6, (VIII. Proposition 2.1), 12, (II section 22, formula (22.4)]). To do this, let us consider a coordinate system $(N; q_1, \ldots, q_n)$ centred at $x \in M$, so that the element

$$t = (\mathrm{d}q_{i_0})_x \odot \cdots \odot (\mathrm{d}q_{i_r})_x \otimes (\mathrm{d}q_{j_2})_x \wedge \cdots \wedge (\mathrm{d}q_{j_k})_x \in S^{r+1}T_x^*(M) \otimes \wedge^{k-1}T_x^*(M)$$

can be identified to

$$j_x^{r+1}(q_{i_0}\ldots q_{i_r}\mathrm{d}q_{j_2} \wedge \ldots \wedge \mathrm{d}q_{j_k}) \in J_x^{r+1}\left(\bigwedge^{k-1} T^*(M)\right).$$

Hence, the element

$$\delta_k^{r-1}(t) = j_x^{r+1}\left(\sum_{h=0}^{r} q_{i_0}\ldots\hat{q}_{i_h}\ldots q_{i_r}\mathrm{d}q_{i_h} \wedge \mathrm{d}q_{j_2} \wedge \cdots \wedge \mathrm{d}q_{j_k}\right)$$

is identified to

$$\sum_{h=0}^{r}(\mathrm{d}q_{i_0})_x \odot \cdots \odot (\widehat{\mathrm{d}q_{i_h}})_x \odot \cdots \odot (\mathrm{d}q_{i_r})_x \otimes (\mathrm{d}q_{i_h})_x \wedge (\mathrm{d}q_{j_2})_x \wedge \cdots \wedge (\mathrm{d}q_{j_k})_x$$

thus proving our claim. Moreover, it follows from the definition of the operators δ_k^{r-1}, that $\delta_k^{r-1} \circ \delta_{k-1}^r = 0$.

In order to prove that $\mathrm{Ker}\ \delta_k^{r-1} \subseteq \mathrm{Im}\ \delta_{k-1}^r$, we proceed by induction on r. For $r = 1$, we have the following commutative diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
S^2T^*M \otimes \bigwedge^{k-1} T^*M & \longrightarrow & T^*M \otimes \bigwedge^k T^*M & \longrightarrow & \bigwedge^{k+1} T^*M \\
\downarrow & & \downarrow & & \downarrow \\
J^2(\bigwedge^{k-1} T^*M) & \xrightarrow{\delta_{k-1}^1} & J^1(\bigwedge^k T^*M) & \xrightarrow{\delta_k^0} & \bigwedge^{k+1} T^*M \\
\downarrow & & \downarrow & & \downarrow \\
J^1(\bigwedge^{k-1} T^*M) & \xrightarrow{\delta_{k-1}^0} & \bigwedge^k T^*M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0. & &
\end{array}$$

The first row is nothing but a part of Koszul's complex so that it is exact. Furthermore, $\delta^0_{k-1}$is surjective as it is checked working in local coordinates; in fact, $\delta^0_{k-1}(j^1_x(q_{i_1}\,\mathrm{d}q_{i_2}\wedge\cdots\wedge\mathrm{d}q_{i_k})) = (\mathrm{d}q_{i_1})_x\wedge\cdots\wedge(\mathrm{d}q_{i_k})_x$. Hence, the second row of the diagram is also exact. For $r\geqslant 2$, we have a commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
S^{r+1}T^*M\otimes\bigwedge^{k-1}T^*M & \longrightarrow & S^rT^*M\otimes\bigwedge^k T^*M & \longrightarrow & S^{r-1}T^*M\otimes\bigwedge^{k+1}T^*M\\
\downarrow & & \downarrow & & \downarrow\\
J^{r+1}(\bigwedge^{k-1}T^*M) & \xrightarrow{\delta^r_{k-1}} & J^r(\bigwedge^k T^*M) & \xrightarrow{\delta^{r-1}_k} & J^{r-1}(\bigwedge^{k+1}T^*M)\\
\downarrow & & \downarrow & & \downarrow\\
J^r(\bigwedge^{k-1}T^*M) & \xrightarrow{\delta^{r-1}_{k-1}} & J^{r-1}(\bigwedge^k T^*M) & \xrightarrow{\delta^{r-2}_k} & J^{r-2}(\bigwedge^{k+1}T^*M)\\
\downarrow & & \downarrow & & \downarrow\\
0 & & 0 & & 0.
\end{array}$$

Again, the first row is exact as it is a part of Koszul's complex and the third row is also exact by virtue of the induction hypothesis. Hence the exactness of the second row can be proved by diagram-chasing. □

Proposition 3. If M is connected, the homomorphism

$$\delta^r_{k-1} : J^{r+1}\left(\bigwedge^{k-1}T^*M\right)\longrightarrow J^r\left(\bigwedge^k T^*M\right)$$

is of constant rank. Accordingly, $\operatorname{Im}\delta^r_{k-1}$ and $\operatorname{Ker}\delta^r_{k-1}$ are vector subbundles of $J^r(\bigwedge^k T^*M)$ and $J^{r+1}(\bigwedge^{k-1}T^*M)$, respectively.

Proof. As M is connected, given two points $x, x'\in M$, there exists a diffeomorphism $\varphi : M\to M$, such that $\varphi(x) = x'$. From the property $\varphi^*(\mathrm{d}\omega_{k-1}) = \mathrm{d}(\varphi^*\omega_{k-1})$, we deduce a commutative diagram

$$\begin{array}{ccc}
J^{r+1}_x(\bigwedge^{k-1}T^*M) & \xrightarrow{\delta^r_{k-1}} & J^r_x(\bigwedge^k T^*M)\\
{\scriptstyle J^{r+1}(\bigwedge^{k-1}\varphi^*)}\uparrow\wr & & \wr\uparrow{\scriptstyle J^r(\bigwedge^k\varphi^*)}\\
J^{r+1}_{x'}(\bigwedge^{k-1}T^*M) & \xrightarrow{\delta^r_{k-1}} & J^r_{x'}(\bigwedge^k T^*M)
\end{array}$$

and since the vertical arrows are isomorphisms, we obtain $\mathrm{rk}_x\ \delta^r_{k-1} = \mathrm{rk}_{x'}\ \delta^r_{k-1}$. □

3.3. The classification theorem

Theorem 4. With the above notations, for every $r\geqslant 2$, we set

$$Z^{r-1}_2(M) = \mathrm{Im}\ \delta^{r-1}_1 = \mathrm{Ker}\ \delta^{r-2}_2 \subseteq J^{r-1}\left(\bigwedge^2 T^*M\right).$$

Then, a Lagrangian $\mathcal{L} : J^r(T^*M)\to\mathbb{R}$ is gauge invariant if and only if it can be written as $\mathcal{L} = \bar{\mathcal{L}}\circ\delta^{r-1}_1$, for a differentiable function $\bar{\mathcal{L}} : Z^{r-1}_2(M)\to\mathbb{R}$.

Proof. The problem reduces to proving that $\mathcal{L}$ is gauge invariant if and only if it is constant on the fibres of δ_1^{r-1}, as $\delta_1^{r-1} : J^r(T^*M) \to Z_2^{r-1}(M)$ is a surjective submersion. Furthermore, the fibres of δ_1^{r-1} are connected as they are affine spaces; in fact, for every one-form ω on M we have

$$(\delta_1^{r-1})^{-1}(j_x^{r-1}(\mathrm{d}\omega)) = (\delta_1^{r-1})^{-1}(0_x) + j_x^r(\omega) \tag{11}$$

where 0_x stands for the zero vector in $J_x^{r-1}(\bigwedge^2 T^*M)$. Hence, it suffices to prove that $\mathcal{L}$ is gauge invariant iff for every δ_1^{r-1}-vertical vector field Y on $J^r(T^*M)$ we have $Y(\mathcal{L}) = 0$. According to formula (6) characterizing gauge invariant Lagrangians, all we need to prove is that the tangent spaces to the fibres of δ_1^{r-1} are spanned by the vector fields

$$Y_\beta = \sum_{i=1}^{n} \frac{\partial}{\partial p^i_{\beta-(i)}} \qquad \forall \beta \in \mathbb{N}^n, 1 \leqslant |\beta| \leqslant r+1.$$

Moreover, as a calculation shows, the local equations for the homomorphism δ_1^{r-1} are as follows:

$$p^{ij}_\beta \circ \delta_1^{r-1} = p^i_{\beta+(j)} - p^j_{\beta+(i)} \qquad i < j,\ 0 \leqslant |\beta| \leqslant r-1.$$

Hence $(\delta_1^{r-1})_*(Y_\beta) = 0$, thus proving that the vector fields Y_β are tangential to the fibres of δ_1^{r-1}. It is not difficult to see that the vector fields Y_β are linearly independent so that

$$\begin{aligned}\mathrm{rk}\{Y_\beta | 1 \leqslant |\beta| \leqslant r+1\} &= \#\{\beta \in \mathbb{N}^n | 1 \leqslant |\beta| \leqslant r+1\}\\ &= \sum_{k=1}^{r+1} \binom{n+k-1}{k} = \binom{n+r+1}{r+1} - 1\end{aligned}$$

and from formula (11) we have

$$\begin{aligned}\dim(\delta_1^{r-1})^{-1}(j_x^{r-1}(\mathrm{d}\omega)) &= \mathrm{rk}(\mathrm{Ker}\ \delta_1^{r-1}) = \mathrm{rk}(\mathrm{Im}\ \delta_0^r)\\ &= \mathrm{rk}\ J^{r+1}(M,\mathbb{R}) - \mathrm{rk}(\mathrm{Ker}\ \delta_0^r)\\ &= \binom{n+r+1}{r+1} - 1.\end{aligned}$$

Hence, the vector fields $\{Y_\beta | 1 \leqslant |\beta| \leqslant r+1\}$ span the tangent spaces to the fibres of δ_1^{r-1}, and the proof is complete. □

3.4. How to calculate $\bar{\mathcal{L}}$

Proposition 5. A Lagrangian $\mathcal{L} : J^r(T^*M) \to \mathbb{R}$ is gauge invariant if and only if $\mathcal{L} \circ j^r(\omega + \mathrm{d}f)$ does not depend on f for every function in $C^\infty(M)$ and every one-form ω on M.

Proof. If $\mathcal{L}$ is gauge invariant, according to the previous theorem we have $(\mathcal{L} \circ j^r(\omega + \mathrm{d}f))(x) = \mathcal{L}(j_x^r(\omega + \mathrm{d}f)) = \bar{\mathcal{L}}(j_x^{r-1}\mathrm{d}\omega)$, thus proving that $\mathcal{L} \circ j^r(\omega + \mathrm{d}f)$ does not depend on the function chosen. The converse is an immediate consequence of the exact sequence

$$J^{r+1}(M,\mathbb{R}) \xrightarrow{\delta_0^r} J^r(T^*M) \xrightarrow{\delta_1^{r-1}} Z_2^{r-1}(M) \to 0$$

which follows from Proposition 2. □

Remark 2. In practice, the above proposition means that in substituting $p^i_\alpha + \partial^{|\alpha|+1} f/\partial q^{\alpha+(i)}$ for p^i_α, $0 \leqslant |\alpha| \leqslant r$, in the arguments of $\mathcal{L}$, the partial derivatives of the arbitrary function f vanish if and only if $\mathcal{L}$ is gauge invariant.

For any indices $i_1, \dots, i_h$, we set $(i_1 \dots i_h) = (i_1) + \cdots + (i_h)$, so that the multi-index $(i_1 \dots i_h)$ depends symmetrically on the indices $i_1, \dots, i_h$; for example $(ij) = (ji)$, $(ijk) = (jki) = (ikj) = \cdots$, etc.

Proposition 6. Let (q_h, p^i_α), $1 \leqslant h \leqslant n$, $1 \leqslant i \leqslant n$, $0 \leqslant |\alpha| \leqslant r$, be the coordinate system on $(p^1)^{-1}_r(N)$ induced from an open coordinate domain $(N; q_h)$ in M, as introduced in section 2.1, formula (1). We set

$$\zeta^{ij}_{(k_1\dots k_h)} = p^i_{(jk_1\dots k_h)} - p^j_{(ik_1\dots k_h)} \qquad \text{if } j > i \leqslant k_1 \leqslant \cdots \leqslant k_h$$

$$\eta^{ij}_{(k_1\dots k_h)} = p^i_{(jk_1\dots k_h)} + p^j_{(ik_1\dots k_h)} \qquad \text{if } j \leqslant i \leqslant k_1 \leqslant \dots \leqslant k_h$$

where $0 \leqslant h \leqslant r-1$, $i, j, k_1, \dots, k_h \in \{1, \dots, n\}$. Then, we have the following.

(1) The functions $(q_h, p_h, \zeta^{ij}_{(k_1\dots k_h)}, \eta^{ij}_{(k_1\dots k_h)})$ are a coordinate system in $J^r(T^*M)$.

(2) Assume a Lagrangian $\mathcal{L} : J^r(T^*M) \to \mathbb{R}$ is gauge invariant. Then, expressed in the above coordinate system, the Lagrangian $\mathcal{L}$ only depends on the variables $q_h, \zeta^{ij}_{(k_1\dots k_h)}$, and this is the local expression for the function $\bar{\mathcal{L}}$ when we substitute $p^{ij}_{(k_1\dots k_h)}$ for $\zeta^{ij}_{(k_1\dots k_h)}$.

Proof. As a simple computation shows, from the above formulae we obtain:

(1) if $j < i \leqslant k_1 \leqslant \cdots \leqslant k_h$, then

$$p^i_{(jk_1\dots k_h)} = \tfrac{1}{2}\left(\zeta^{ij}_{(k_1\dots k_h)} + \eta^{ji}_{(k_1\dots k_h)}\right)$$

(2) if $i = j \leqslant k_1 \leqslant \cdots \leqslant k_h$, then

$$p^i_{(ik_1\dots k_h)} = \tfrac{1}{2}\eta^{ii}_{(k_1\dots k_h)}$$

(3) if $j > i < k_1 \leqslant \cdots \leqslant k_h$, then

$$p^i_{(jk_1\dots k_h)} = \eta^{ij}_{(k_1\dots k_h)} - \tfrac{1}{2}\eta^{k_1 j}_{(k_2\dots k_l i k_{l+1}\dots k_h)} - \tfrac{1}{2}\zeta^{jk_1}_{(k_2\dots k_l i k_{l+1}\dots k_h)}$$

where $1 \leqslant l \leqslant h$ is the least integer such that $j < k_1 \leqslant \cdots \leqslant k_l \leqslant i \leqslant k_{l+1} \leqslant \dots \leqslant k_h$;

(4) if $j > i = k_1 \leqslant \cdots \leqslant k_h$, then

$$p^i_{(jk_1\dots k_h)} = \eta^{ij}_{(k_1\cdots k_h)} - \tfrac{1}{2}\eta^{jj}_{(k_2\cdots k_l i k_{l+1}\cdots k_h)}$$

where l is defined as in item (3).

This proves the first part of the statement. The second part follows directly from the definitions and our hypothesis. □

Example 7. There is a general procedure in order to obtain quadratic gauge invariant Lagrangians of arbitrary order. Let g be a pseudo-Riemannian metric on a manifold M. For every $x \in M$, the metric induces an isomorphism $\flat : T_xM \to T^*_xM$ given by $X^\flat(Y) = g(X, Y)$. The inverse of $\flat$ is usually denoted by $\sharp : T^*_xM \to T_xM$, and we can define a non-degenerate quadratic form (still denoted by g) on T^*_xM by setting $g(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp)$. More generally, we can define a metric on $\bigwedge^r T^*M$ by setting

$$g(\alpha_1 \wedge \cdots \wedge \alpha_r, \beta_1 \wedge \cdots \wedge \beta_r) = \det(g(\alpha_i, \beta_j))^r_{i,j=1}.$$

Finally, we can define a metric on $S^kT^*M \otimes \bigwedge^r T^*M$ as

$$g(\alpha_1 \odot \cdots \odot \alpha_k \otimes \omega_r, \beta_1 \odot \cdots \odot \beta_k \otimes \omega'_r) = \sum_{\pi \in \mathrm{Perm}_k} \sum_{i=1}^{k} g(\alpha_{\pi(i)}, \beta_{\pi(i)}) g(\omega_r, \omega'_r).$$

Let ∇ be the Levi–Civita connection of g, and let

$$\rho_r : J^r\left(\bigwedge^2 T^*M\right) \to S^r T^*M \otimes \bigwedge^2 T^*M$$

be the retract of the natural injection of $S^r T^*M \otimes \bigwedge^2 T^*M$ into $J^r(\bigwedge^2 T^*M)$ in the exact sequence of vector bundles over M,

$$0 \to S^r T^*M \otimes \bigwedge^2 T^*M \to J^r\left(\bigwedge^2 T^*M\right) \to J^{r-1}\left(\bigwedge^2 T^*M\right) \to 0$$

given by $\rho_r(j^r_x\omega_2) = \mathrm{sym}(\nabla^r\omega_2)_x$, where sym stands for the symmetrization operator. By means of ρ_r, we can thus obtain a splitting

$$J^r\left(\bigwedge^2 T^*M\right) \cong J^{r-1}\left(\bigwedge^2 T^*M\right) \oplus S^r T^*M \otimes \bigwedge^2 T^*M.$$

Inductively, for every $r \geqslant 0$, we can define a metric on$J^r(\bigwedge^2 T^*M)$ by the formula

$$g(j^r_x\omega_2, j^r_x\omega_2') = g(j^{r-1}_x\omega_2, j^{r-1}_x\omega_2') + g(\rho_r(j^r_x\omega_2), \rho_r(j^r_x\omega_2')).$$

According to Theorem 4, we thus have a gauge invariant Lagrangian

$$\mathcal{L}_r : J^r(T^*M) \to \mathbb{R}$$

by setting $\mathcal{L}_r(j^r_x\omega) = \frac{1}{2}g(j^{r-1}_x(\mathrm{d}\omega), j^{r-1}_x(\mathrm{d}\omega))$ Notice that $\mathcal{L}_r = \bar{\mathcal{L}}_r \circ \delta^{r-1}_1$, where $2\bar{\mathcal{L}}_r$ is the norm of the metric induced on $J^{r-1}(\bigwedge^2 T^*M)$, as explained above.

Let us now consider the particular case $M = \mathbb{R}^2$ with the standard Lorentzian metric $g = \mathrm{d}q_0^2 - \mathrm{d}q_1^2$. First we shall compute the local expression for $\mathcal{L}_2$. Let $\omega = f_0\,\mathrm{d}q_0 + f_1\,\mathrm{d}q_1$ be an arbitrary one-form on M. In our case we have

$$\mathcal{L}_2(j^2_x\omega) = \tfrac{1}{2}g(j^1_x(\mathrm{d}\omega), j^1_x(\mathrm{d}\omega)) = \tfrac{1}{2}\{g(\mathrm{d}\omega, \mathrm{d}\omega)(x) + g(\nabla\,\mathrm{d}\omega, \nabla\,\mathrm{d}\omega)(x)\}.$$

Hence,

$$\mathcal{L}_2 = \tfrac{1}{2}\{(p^0_{(11)} - p^1_{(01)})^2 - (p^0_{(01)} - p^1_{(00)})^2 - (p^0_1 - p^1_1)^2\}$$

and the corresponding Euler–Lagrange equations are

$$\begin{gathered}
\frac{\partial^2 f_0}{\partial q_1^2} - \frac{\partial^2 f_1}{\partial q_0 \partial q_1} - \frac{\partial^2 f_0}{\partial q_0^2 \partial q_1^2} + \frac{\partial^2 f_1}{\partial q_0^3 \partial q_1} + \frac{\partial^2 f_0}{\partial q_1^4} - \frac{\partial^2 f_1}{\partial q_0 \partial q_1^3} = 0 \\
-\frac{\partial^2 f_0}{\partial q_0 \partial q_1} + \frac{\partial^2 f_1}{\partial q_0^2} + \frac{\partial^2 f_0}{\partial q_0^3 \partial q_1} - \frac{\partial^2 f_1}{\partial q_0^4} - \frac{\partial^2 f_0}{\partial q_0 \partial q_1^3} + \frac{\partial^2 f_1}{\partial q_0^2 \partial q_1^2} = 0.
\end{gathered}$$

As a simple (but rather long) computation shows, the above equations can be intrinsically written as

$$(\delta\mathrm{d} + (\delta\mathrm{d})^2)(\omega) = 0$$

where δ stands for the codifferential operator.

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